Journal of Statistical Physics, Vol. 120, Nos. 3/4, August 2005 (© 2005)
DOI: 10.1007/s10955-005-5478-7

Anomalous Pulsation

Iddo Eliazar[1] and Joseph Klafter[2]

Received November 16, 2004; accepted April 21, 2005

Subordinating regular diffusion – namely, Brownian motion – to random time flows generated by Lévy noises may result in anomalous diffusion. Motivated by this phenomena, and by the recent interest in the phenomena of *blinking* in various physical systems, we explore the subordination of regular stochastic pulsation – namely, Poisson process – to random time flows generated by Lévy noises. We show that such subordination may yield, analogous to the case of diffusion, *anomalous pulsation*. Anomalous pulsation displays the following anomalous behaviors, which are impossible in the case of regular pulsation: (i) simultaneous emission of multiple pulses; (ii) non-linear local pulsation rates; (iii) clustering of pulsation epochs.

KEY WORDS: Blinking phenomena; anomalous pulsation; time-to-pulsation; inter-pulsation period; pulsation multiplicity; Lévy noises and processes; Ornstein–Uhlenbeck rates; moving-average rates.

1. INTRODUCTION

Random pulsation sources are ubiquitous in the 'real world' surrounding us. In particular, in recent years there is a growing interest in the phenomena of *blinking* in a broad range of physical systems such as: individual semiconductor nanocrystals (quantum dots);[1] fluctuating single enzymes;[2,3] and, single pores translocating biomolecules.[4] All these systems exhibit, for different physical reasons, time series which correspond to fluorescent and non-fluorescent periods – often referred to as "on" and "off" periods – or to periods of finite and zero current. These "on" and

[1]Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel; e-mail: eliazar@math.biu.ac.il; eliazar@post.tau.ac.il
[2]School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: klafter@post.tau.ac.il

"off" periods have been found, in some cases, to display *Lévy*-type statistics. The aim in the study of the blinking phenomena is to obtain information, mainly dynamical, on the underlying physical systems, and on how they are driven.

Motivated by these recent observations, we consider a stochastic pulsation model – driven by an arbitrary Lévy noise source – that captures some of the characteristics of the time series generated by 'blinking systems'. Our goal in this work is to establish a mathematical framework that will serve scientists to model particular physical blinking systems, and to analyze the time series generated by them. For reasons to be described below we call this framework *Anomalous Pulsation*.

Rigorously speaking, a pulsation process is a *point process* on the non-negative half-line.[5] That is, a random countable collection of points – the 'pulsation epochs' – scattered stochastically along the time axis $[0, \infty)$.

The simplest and most basic mathematical model of a random pulsation source is the *Poisson process*, in which the inter-pulsation periods are independent and exponentially distributed. The exponential distribution, governing the length of the inter-pulsation periods, has various unique features: (i) it attains maximum entropy (amongst all distributions on the positive half-line with a given mean); (ii) it renders the pulsation epochs unpredictable: at any given time point t, the next pulsation epoch can not be predicted based on the history of pulsations up to time t; and, most important; (iii) it renders the pulsation process *Markovian* and *Lévy*.

Lévy motions are continuous-time random processes with stationary and independent increments (which are continuous in probability). Lévy motions constitute the most fundamental family of continuous-time random motions. Since their introduction in the 1930s,[6–8] Lévy motions were studied and researched extensively by both theoreticians and applied scientists. The literature on Lévy motions is vast, and their range of applications encompasses numerous fields of science and engineering. See refs. 9–12 for the theory of Lévy processes, and refs. 13–19 and references therein for their applications. The best known examples of the Lévy family are the Poisson process (random pulsations), and Brownian motion (random continuous motion).

For long years Brownian motion served as the dominant 'process-of-choice' for the modeling of diffusive motion of particles.[20,21] However, in recent years more and more attention was given to the study of anomalous motion – both sub-diffusive and super-diffusive – conducted in various physical systems, which Brownian motion fails to adequately model.[17,22] Nevertheless, anomalous diffusions are obtained by the *subordination* of regular diffusion – i.e., Brownian motion – to *random time flows*.[23]

Temporal subordination arises naturally in systems whose subjective 'operational time' is different from the objective 'physical time'. That is, systems which tick according to an internal – often stochastic and irregular – 'subjective clock', rather than pace according to the universal 'objective clock' (whose time flow is deterministic and linear). Furthermore, temporal subordination is a most effective mean of introducing anomalies into diffusive processes *without* distorting or changing their underlying transport mechanisms. In particular, subordination elegantly produces both sub-diffusive and super-diffusive motions from regular diffusive motions such as Random Walks and Brownian motion. (see, for example, refs. 24–27).

Motivated by the example of anomalous diffusions generated by the subordination of Brownian motion to random time flows, we consider an analogous framework for random pulsations. Namely, we study the subordination of regular stochastic pulsation – i.e., Poisson process – to random time flows, and explore the resulting pulsation process.

Consider, for starters, a unit-rate Poisson process $N=(N(t))_{t\geqslant 0}$. That is, N is a Poisson process whose inter-pulsation periods are exponentially distributed with unit mean. Applying the simple linear temporal transformation $t \to \lambda t$ (λ being a positive constant) transforms the unit-rate Poisson process N to a λ-rate Poisson process $Y=(Y(t))_{t\geqslant 0}$ given by

$$Y(t) := N(\lambda t) = N\left(\int_0^t \lambda du\right). \tag{1}$$

Observing the right-hand-side of Eq. (1) implies that the rate λ is, in fact, the 'speed of time'.

Now, rather than taking the 'speed of time' to be constant, consider the case where it is a non-negative valued and stationary random process $\Lambda=(\Lambda(t))_{t\geqslant 0}$, which is independent of the Poisson process N. This yields a *pulsation process* $Y=(Y(t))_{t\geqslant 0}$ given by

$$Y(t) = N\left(\int_0^t \Lambda(u) du\right). \tag{2}$$

Such processes were first introduce by Cox,[28] calling them 'doubly stochastic Poisson processes' (see also refs. 29 and 30). We shall henceforth refer to Λ as the *rate process* of the pulsation process Y. Since we wish to fucus on blinking systems displaying Lévy-type statistics, the rate process Λ is taken to be driven by an underlying *one-sided* Lévy motion $X=(X(t))_{t\geqslant 0}$.

Our aim is to study the emission statistics of the resulting pulsation process Y, and compare its behavior to that of the 'standard' Poisson process N. As we shall demonstrate, the following anomalous behaviors may be exhibited:

- **Linear *vs* Non-Linear Rates.** For the Poisson process N the local rate of pulsation is *linear*. That is, the probability that an emission would occur during an arbitrary time interval $(t, t+\delta)$ is - as $\delta \to 0$ - linear in the interval length δ. For the pulsation process Y the analogous probability may be *non-linear* in the interval length δ. Namely; as $\delta \to 0$, the probability of pulsing during the time interval $(t, t+\delta)$ may be linear in δ^{α}, where the exponent α is in the range $0 < \alpha < 1$.
- **Spacing *vs* Clustering.** The jumps of the Poisson process N are temporally spaced – the length of the spacing being exponentially distributed (with unit mean). That is, for the Poisson process N the notion of 'inter-pulsation periods' is well defined. For the pulsation process Y this notion may, however, be ill-defined. Specifically; given that Y pulsed at time t, the time till the next pulsation epoch may equal zero – rendering the notion of 'inter-pulsation periods' undetermined. Using a deterministic analogy; the 'pulsation epochs' of the Poisson process N topologically resemble an arithmetic sequence, whereas the 'pulsation epochs' of the pulsation process Y topologically resemble a Cantor-like set.
- **Single *vs* Multiple Pulsations.** The jumps of the Poisson process N are always of unit size – implying that the process emits *single* pulses. The pulsation process Y may, however, have jumps of random integer-valued size. That is, the pulsation process Y may emit *multiple simultaneous pulses* – the 'pulsation multiplicity' being a random integer (governed by some probability law). Pictorially; the 'jump heights' of the Poisson process N are constant and of unit size, whereas the 'jump heights' of the pulsation process Y vary randomly.

Realizations of an anomalous pulsation process are simulated in Figures 1, 2, and 3 below. In these simulations the rate process was taken to be a stationary Ornstein–Uhlenbeck process driven by a selfsimilar ('fractal') one-sided Lévy motion (see Section 5 for the details). The resulting realizations well exemplify the anomalous phenomena of *clustering* and *multiple pulsations*.

The manuscript is organized as follows. We begin, in Section 2, with the case where the rate process Λ is a Lévy noise – the derivative of the underlying one-sided Lévy motion X. In this case the Poisson process N turns out to be subordinated to the process X, and the resulting pulsation

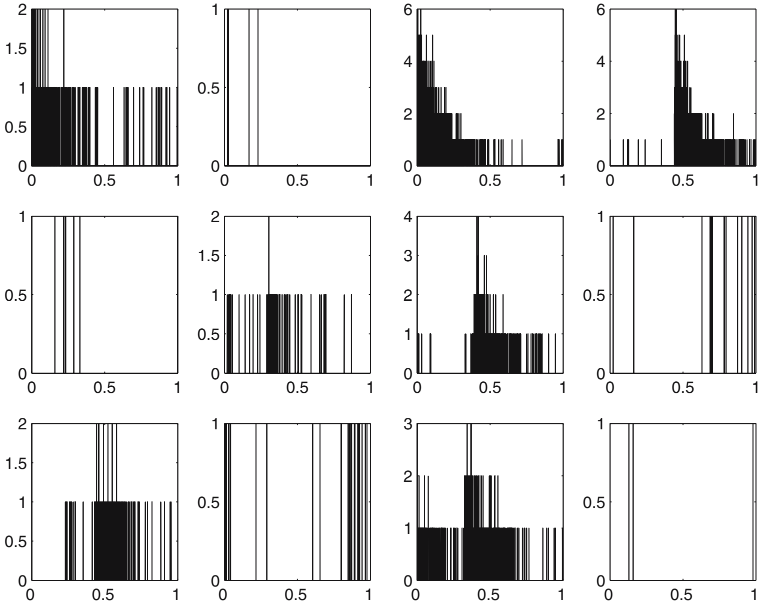

Fig. 1.

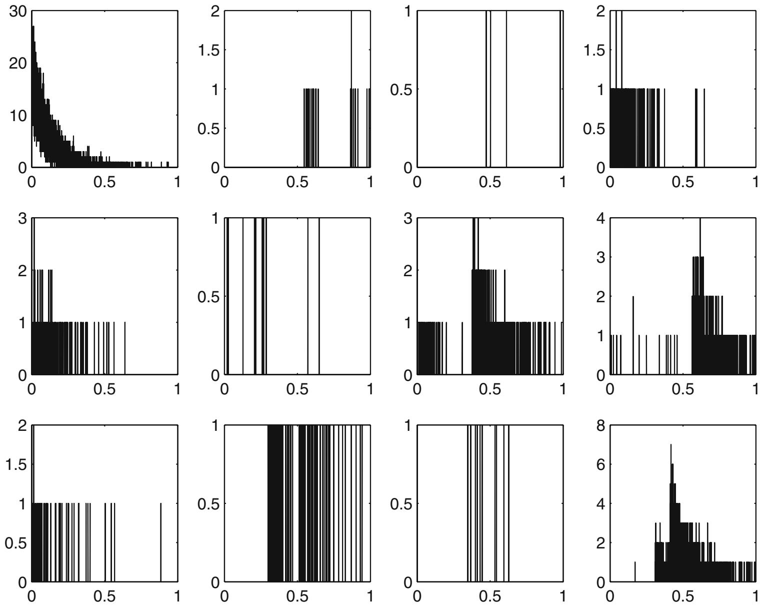

Fig. 2.

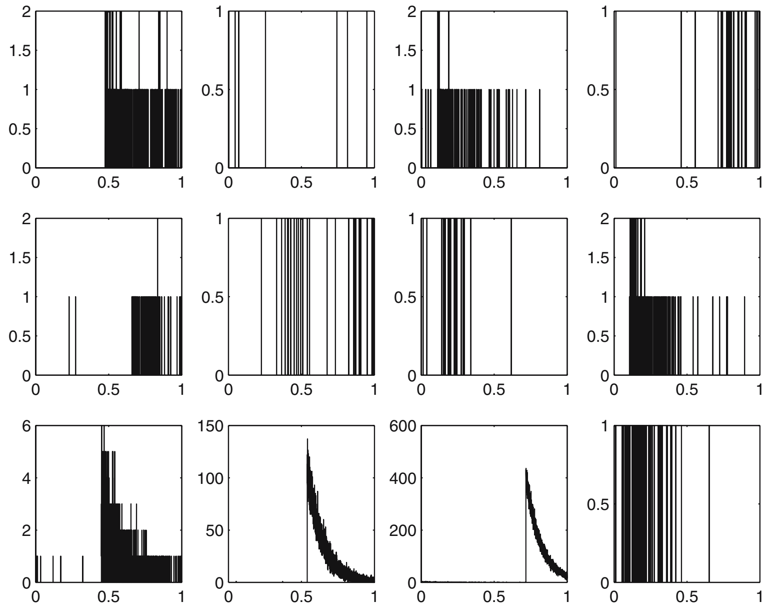

Fig. 3.

Figs. 1, 2, 3. In the figures depicted above we simulate realizations of an anomalous pulsation process with a stationary Ornstein–Uhlenbeck rate. Each figure contains 12 different simulations of the anomalous pulsation process along a unit-long time interval (the x axis). Pulsations are represented by vertical bars – the height of the bars representing the pulsation multiplicity (y axis).

In the simulations the Ornstein–Uhlenbeck parameter κ equals 1, and the driving Lévy motion is governed by the Lévy characteristic $\phi(\omega) = 10 \cdot \omega^{2/5}$ (namely; the driving Lévy motion is 'fractal' with selfsimilarity index $\alpha = 2/5$). The underlying 'fractal' Lévy noise was generated using the Chambers-Mallows-Stuck simulation algorithm.[39]

The theory developed predicts this anomalous pulsation process to display both *clustering* and *multiple pulsations* – which are indeed vividly exhibited by the simulations. Note how a huge 'surge' in the driving Lévy motion cascades into an 'avalanche' of clustered multiple pulsations, and how the Ornstein–Uhlenbeck decay causes the 'avalanche' to dissipate exponentially (this is best exemplified by the simulations in the bottom line of Figure 3).

process Y is given by $Y(t) = N(X(t))$. We show that this subordination always produces multiple pulsations, and study in detail Y's 'pulsation multiplicity'. In Section 3 we review the first and second order statistics – mean, variance, auto-correlation, and power spectrum – of general pulsation processes Y defined by Eq. (2). In Section 4 we turn to study the emission structure and emission statistics of these general processes: (i) the *cumulative pulsations* during a given time interval; (ii) the *time-to-pulsation* elapsing from a given time epoch till the first pulsation epoch occurring after it; (iii) the *inter-pulsation period* between consecutive pulsation epochs; (iv) the *pulsation multiplicity* of the emissions. In particular, we show that there is a fundamental qualitative difference between the case of rate processes Λ having finite mean, and the case of rate processes Λ having infinite mean. Equipped with the general results of Section 4 we turn to explore pulsation processes with *Moving-Average* rates, i.e., rate processes Λ which are moving averages of the underlying Lévy motion X. In Section 5 we study the case of exponentially-decaying moving averages – yielding *Ornstein–Uhlenbeck* rate processes. In section 6 we study general Moving-Average rates.

A note about notations Throughout the manuscript: $\mathbf{P}(\cdot)$ = Probability, and $\mathbf{E}[\cdot]$ = Expectation. Also, PGF is a shortcut for *Probability Generating Function*, and IID is a shortcut for *Independent and Identically-Distributed.*

2. LÉVY-NOISE RATES

In this section we explore the case where the stochastic rate process $\Lambda=(\Lambda(t))_{t\geqslant 0}$ is a Lévy noise. That is, $\Lambda(t)=\dot{X}(t)$ where $X=(X(t))_{t\geqslant 0}$ is a one-sided Lévy motion.[11] We denote by $\phi(\omega)$ ($\omega\geqslant 0$) the Lévy characteristic of the motion X, i.e.,

$$\mathbf{E}[\exp\{-\omega X(t)\}]=\exp\{-\phi(\omega)t\}.$$

Since $\Lambda(t)=\dot{X}(t)$ we have $\int_0^t \Lambda(s)ds=X(t)$, and hence the representation (2) of the pulsation process $Y=(Y(t))_{t\geqslant 0}$ becomes

$$Y(t)=N(X(t)). \tag{3}$$

That is, Y is given by the *subordination* of the Poisson process N to the one-sided Lévy motion X.

2.1. Cumulative Pulsation and Time-to-Pulsation

Since both N and X are independent Lévy motions, the subordination (3) implies that the resulting pulsation process Y is, too, a Lévy motion (see, for example, ref. 11). Furthermore, the PGF of the increments of Y is given by ($|z|\leqslant 1$):

$$\mathbf{E}[z^{Y(s+t)-Y(s)}]=\exp\{-\phi(1-z)t\}. \tag{4}$$

The proof of Eq. (4) follows straightforwardly using conditioning (and the fact that N is a unit-rate Poisson process, and that X is a one-sided Lévy motion with Lévy characteristic $\phi(\cdot)$):

$$\begin{aligned}\mathbf{E}[z^{Y(s+t)-Y(s)}] &= \mathbf{E}\left[\mathbf{E}[z^{N(X(s+t))-N(X(s))}|X]\right]\\ &= \mathbf{E}\left[\exp\{-(1-z)(X(s+t)-X(s))\}\right] = \exp\{-\phi(1-z)t\}.\end{aligned}$$

Consider now an observation of the pulsation process initiated at an arbitrary time point s, and let $\tau(s)$ denote the length of the time period, elapsing from time s, till the first pulse is observed:

$$\tau(s) = \inf\{t \geqslant 0 | Y(s+t) - Y(s) > 0\}.$$

What is the distribution of $\tau(s)$?

Well, $\tau(s)$ turns out to be independent of s, and is *exponentially distributed* with parameter $\phi(1)$. That is ($t \geqslant 0$):

$$\mathbf{P}(\tau(s) > t) = \exp\{-\phi(1)t\}. \tag{5}$$

Equation (5) is an immediate consequence of Eq. (4), as the following calculation shows:

$$\begin{aligned}\mathbf{P}(\tau(s) > t) &= \mathbf{P}(Y(s+t) - Y(s) = 0)\\ &= \mathbf{E}[z^{Y(s+t)-Y(s)}]\big|_{z=0} = \exp\{-\phi(1)t\}.\end{aligned}$$

Hence, we obtained that the 'time-to-pulsation' – no matter when the observation was initiated – is exponentially distributed with parameter $\phi(1)$. Using standard results from the theory of renewal processes,[31] this fact implies that the inter-pulsation periods are also exponentially distributed with parameter $\phi(1)$, and are IID. We shall prove this result below, using an alternative 'Lévy-based' approach.

2.2. Pulsation Multiplicity

In the previous section, we deduced that the inter-pulsation periods are IID and exponentially distributed with parameter $\phi(1)$. Hence, the pulsation epochs of the process Y follow a Poisson process with rate $\phi(1)$. This, in turn, implies that the pulsations of the process Y must be *multiple* – since otherwise we shall arrive at a contradiction. Indeed, if the pulsations of Y are single, then Y is a Poisson process with rate $\phi(1)$ and hence the PGF of its increments is

$$\mathbf{E}[z^{Y(s+t)-Y(s)}] = \exp\{-\phi(1)(1-z)t\}$$

– in sharp contradiction to Eq. (4).

Hence, unlike the underlying Poisson process N (which emits single pulses), the pulsation process Y must emit *multiple simultaneous pulses*. In other words, when Y pulses – it pulses with a vengeance (!). Our task now is to compute the multiplicity of the emissions of the process Y.

We define the *pulsation multiplicity* M as the limit, in law, as $\delta \to 0$, of $Y(\delta)$ *conditioned* on the event $\{Y(\delta) > 0\}$. The PGF of the pulsation multiplicity M is hence given by

$$\mathbf{E}[z^M] = \lim_{\delta \to 0} \mathbf{E}[z^{Y(\delta)} | Y(\delta) > 0]. \tag{6}$$

Using Lemma 1 (see section A.1 of the appendix) and the PGF of the cumulative pulsation (Eq. (4)), the right-hand-side of (6) is straightforwardly computed yielding

$$\mathbf{E}[z^M] = 1 - \frac{\phi(1-z)}{\phi(1)}. \tag{7}$$

Expanding both sides of (7) into power series in z and equating the coefficients, we obtain that the probability frequencies of the pulsation multiplicity M are given by

$$\mathbf{P}(M = m) = \frac{(-1)^{m-1}}{m!} \frac{\phi^{(m)}(1)}{\phi(1)} \tag{8}$$

(where $m = 1, 2, \ldots$, and where $\phi^{(m)}$ denotes the mth derivative of the Lévy characteristic ϕ).

Furthermore, Eq. (7) implies that the 'output' pulsation process Y will have *single* pulsations if and only if the subordinating 'input' process X is a *degenerate* (i.e., deterministic) Lévy motion. Indeed:

$$\begin{aligned} & Y \text{ emits single pulses} \\ \Leftrightarrow \quad & \text{the right-hand-side of (7) equals } z \\ \Leftrightarrow \quad & \phi(\omega) \equiv \phi(1)\omega \\ \Leftrightarrow \quad & X(t) \equiv \phi(1)t. \end{aligned}$$

Combining the result regarding the inter-pulsation periods (obtained in the previous subsection) together with the result regarding the pulsation multiplicity (obtained above), we can conclude that: the pulsation process Y is a *compound Poisson process* with rate $\phi(1)$, and integer-valued jumps whose sizes are governed by the probability frequencies (8).

2.3. From Subordination to Compounding

We now show directly, using a 'Lévy-based' approach, that the subordination of the Poisson process N to the one-sided Lévy motion X is *equivalent* to compounding it.

Consider the compound Poisson process $Z=(Z(t))_{t\geqslant 0}$ (generated from the Poisson process N) defined by

$$Z(t)=\sum_{n=1}^{N(\lambda t)} J_n, \tag{9}$$

where λ is a positive constant and where $\{J_n\}_{n=1}^{\infty}$ is an IID sequence of integer-valued random variables (independent of the Poisson process N). In other words, Z is a compound Poisson process with rate λ and integer-valued jumps of size J. We denote by $G(z)=\mathbf{E}[z^J]$, $|z|\leqslant 1$, the PGF of the jump-size J. The compound Poisson process Z is an integer-valued Lévy motion, and the PGF of its increments is given by

$$\mathbf{E}[z^{Z(s+t)-Z(s)}]=\exp\{-\lambda(1-G(z))t\} \tag{10}$$

(the proof of Eq. (10) is analogous to the proof of Eq. (4)).

Now, since both the subordinated process Y and the compound process Z are integer-valued Lévy motions, they are equal, in law, if and only if their characteristic PGFs (given, respectively, in Eq. (4) and (10)) are equal. That is, if and only if

$$\phi(1-z)=\lambda(1-G(z)). \tag{11}$$

Hence, in order that the compound Poisson process Z be equal, in law, to the subordinated process Y we need that: **(i)** $\lambda=\phi(1)$ (taking $z=0$ in Eq. (11) and noting that $G(0)=\mathbf{P}(J=0)=0$); and, **(ii)**

$$G(z)=1-\frac{\phi(1-z)}{\phi(1)} \tag{12}$$

(substituting $\lambda=\phi(1)$ back into Eq. (11)).

We have thus obtained, *simultaneously*, the following pair of facts (deduced separately in the two previous subsections): **(i)** the inter-pulsation periods of the process Y are exponentially-distributed with parameter $\phi(1)$; and, **(ii)** the pulsations of Y are multiple, and their multiplicity is governed by the PGF given in equation (12).

2.4. The Lévy–Khinchin Representation

The reason for the multiple pulsations of Y stems from the *discontinuous* structure of the subordinating Lévy rate process X. One-sided Lévy motions are pure-jump processes, and their evolution (growth) is purely discontinuous.[27] These jumps/discontinuities in the trajectory of the 'input' process X are the cause for the multiple pulsations of the 'output' process Y. We shall now quantitatively formalize this qualitative assertion.

The exact jump structure of Lévy motions is specified by the celebrated *Lévy–Khinchin formula* (see, for example, ref. 11), asserting that: if $X=(X(t))_{t\geqslant 0}$ is a one-sided Lévy motion then its Lévy characteristic $\phi(\omega)$ $(\omega\geqslant 0)$ admits the integral representation

$$\phi(\omega)=\int_0^\infty (1-\exp\{-\omega x\})\,\mu(dx), \tag{13}$$

where $\mu(\cdot)$ is a measure (on the non-negative half-line) satisfying the integrability condition $\int_0^\infty \min\{x,1\}\mu(dx)<\infty$. The measure $\mu(\cdot)$ is called the *Lévy measure* of the Lévy motion X. Informally, the Lévy–Khinchin formula (13) asserts that X is a *continuum Poisson superposition*, where jumps of size x occur at rate $\mu(dx)$.

Differentiating both sides of Eq. (13) m times gives

$$\phi^{(m)}(1)=(-1)^{m-1}\int_0^\infty \exp\{-x\}x^m\,\mu(dx).$$

Substituting this into Eq. (8) yields the following Lévy–Khinchin representation of the probability frequencies of the pulsation multiplicity:

$$\mathbf{P}(M=m)=\frac{1}{\phi(1)}\cdot\frac{1}{m!}\int_0^\infty \exp\{-x\}x^m\,\mu(dx) \tag{14}$$

$(m=1,2,\ldots)$.

Equation (14) establishes an explicit correspondence between the jump structure of the 'input' Lévy rate process X (via its Lévy measure $\mu(\cdot)$), and the jumps of the 'output' pulsation process Y (via its pulsation multiplicity M). The second term on the right-hand-side of (14) is rather interesting – it is the ratio of a 'Lévy Gamma function' to the 'standard Gamma function' (both evaluated at the point $m+1$):

$$\frac{\int_0^\infty \exp\{-x\}x^m\,\mu(dx)}{\int_0^\infty \exp\{-x\}x^m\,dx}.$$

2.5. Examples

We present a few examples of 'input' Lévy rate processes (X) and their resulting pulsation multiplicities (M). The examples are computed using Eq. (14). For further details regarding the one-sided Lévy processes used as 'input' we refer the readers to ref. 11. In the examples below η and ν are positive parameters and $q := 1/(1+\eta)$.

1. If X is a compound Poisson process with exponentially distributed jumps then its Lévy measure is of the form $\mu(dx) = \exp\{-\eta x\}dx$ and hence

$$\mathbf{P}(M = m) = \eta q^m.$$

In other words, M is a *Geometric* random variable with parameter $1-q$.

2. If X is compound Poisson with Gamma-distributed jumps then its Lévy measure is of the form $\mu(dx) = \exp\{-\eta x\}x^{\nu-1}\,dx$ and hence

$$\mathbf{P}(M = m) = c\frac{\Gamma(m+\nu)}{m!}q^m \underset{m\to\infty}{\sim} cq^m m^{\nu-1},$$

where $1/c = \Gamma(\nu)\,((1+1/\eta)^\nu - 1)$.

3. If X is a Gamma process – i.e., a Lévy motion with Gamma-distributed increments – then its Lévy measure is of the form $\mu(dx) = \exp\{-\eta x\}x^{-1}\,dx$ and hence

$$\mathbf{P}(M = m) = c\frac{q^m}{m},$$

where $1/c = \ln\{1 + 1/\eta\}$. Furthermore, the Lévy characteristic of X is $\phi(\omega) = \ln\{1 + \omega/\eta\}$ and hence (using Eq. (4)) the PGF of the cumulative pulsation is

$$\mathbf{E}[z^{Y(t)}] = \left(\frac{1-q}{1-qz}\right)^t.$$

This, in turn, implies that for $t = 1, 2, \ldots$ the random variable $Y(t) + t$ is *Negative Binomial* with parameters $(t, 1-q)$.

4. If the Lévy measure of X is of the form $\mu(dx) = \exp\{-\eta x\} x^{-1-\nu}\, dx$ $(0 < \nu < 1)$ then

$$\mathbf{P}(M = m) = c \frac{\Gamma(m-\nu)}{m!} q^m \underset{m\to\infty}{\sim} c \frac{q^m}{m^{1+\nu}},$$

where $1/c = (\Gamma(1-\nu)/\nu)\,(1-(\eta q)^\nu)$.

5. The last, and most important example – as we shall yet see in the sequel – is that of *selfsimilar* Lévy rate processes.[32] If X is selfsimilar of order ν $(0 < \nu < 1)$ then its Lévy measure is of the form $\mu(dx) = x^{-1-\nu}\, dx$ and hence: the PGF of the pulsation multiplicity is

$$\mathbf{E}[z^M] = 1 - (1-z)^\nu; \tag{15}$$

the probability frequencies are

$$\mathbf{P}\,(M = m) = \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(m-\nu)}{m!} \underset{m\to\infty}{\sim} \frac{\nu}{\Gamma(1-\nu)} \frac{1}{m^{1+\nu}}; \tag{16}$$

and, the asymptotic behavior of the probability tails is given by

$$\mathbf{P}\,(M > m) \underset{m\to\infty}{\sim} \frac{1}{\Gamma(1-\nu)} \frac{1}{m^\nu}. \tag{17}$$

Furthermore, the Lévy characteristic of X is $\phi(\omega) = (\Gamma(1-\nu)/\nu)\,\omega^\nu$ and hence (using Eq. (4)) the PGF of the cumulative pulsation is

$$\mathbf{E}[z^{Y(t)}] = \exp\left\{-\,(t\Gamma(1-\nu)/\nu)\,(1-z)^\nu\right\}.$$

This last equation – using Karamata's Tauberian theorem for random variables (see, for example, ref. 33) – implies that the asymptotic behavior of the probability tails of the cumulative pulsations is given by

$$\mathbf{P}\,(Y(t) > y) \underset{y\to\infty}{\sim} \frac{t}{\nu} \frac{1}{y^\nu}.$$

Note that in all the examples above the functional form of the probability frequencies turned out to be *asymptotically identical* to the functional form of the density of the Lévy measure. Namely, the probability frequency $\mathbf{P}\,(M = m)$ (as a function of the variable m) 'inherited', asymptotically, the functional form of the density $\mu(dx)/dx$ (as a function of the variable x).

3. STATIONARY RATE PROCESSES: INTER-DEPENDENCE OF THE PULSATION PROCESS

When the rate process Λ is a Lévy-noise, the increments of the pulsation process Y – as we have shown in the previous section – are both stationary and independent. As indicated in the introduction, are aim however is to explore the case of pulsation processes excited by general stationary rate processes which are driven by Lévy noises. The stationarity of the underlying rate process Λ implies the stationarity of the increments of pulsation process Y. However, the increments of Y need not be – and in general are not – independent.

In this section, we study the inter-dependence of the pulsation process Y, excited by an arbitrary stationary rate process Λ. We begin with the computation of the mean, variance, and covariance of Y, and then continue on to analyze the auto-correlation function and power spectrum of Y.

3.1. Mean, Variance, and Covariance

Let λ and $C(\cdot)$ denote, respectively, the mean and auto-covariance function of the underlying stationary rate process Λ ($0 \leqslant s, t < \infty$):

$$\lambda = \mathbf{E}[\Lambda(t)] \tag{18}$$

$$C(t-s) = \mathbf{Cov}(\Lambda(t), \Lambda(s)) . \tag{19}$$

Given a time interval I we use the shorthand $Y(I)$ to denote the cumulative pulsations during I, and denote by $|I|$ the length of I (namely; if $I=(a,b)$ then $Y(I) = Y(b) - Y(a)$ and $|I| = b-a$).

Let I and J be arbitrary time intervals. The mean and variance of $Y(I)$, and the covariance of $Y(I)$ and $Y(J)$, are given by:

$$\mathbf{E}[Y(I)] = \lambda |I|, \tag{20}$$

$$\mathbf{Var}(Y(I)) = \lambda |I| + \int_I \int_I C(t-s) dt\ ds, \tag{21}$$

and,

$$\mathbf{Cov}(Y(I), Y(J)) = \lambda |I \cap J| + \int_I \int_J C(t-s) dt\ ds. \tag{22}$$

Results (20)–(22), in various formulations, are well known (see, for example, refs. 29 and 30). A proof of Eqs. (20)–(22) is given in the appendix (see section A.2).

3.2. Auto-Correlation

Given a positive constant Δ, we define the auto-correlation function $\rho_\Delta(T)$, $T \geqslant \Delta$, as follows:

$$\rho_\Delta(T) := \mathbf{Cov}(Y((T, T+\Delta)), Y((0, \Delta)))\,. \tag{23}$$

That is, $\rho_\Delta(T)$ measures the covariance of the cumulative pulsations during the time intervals $(0, \Delta)$ and $(T, T+\Delta)$. Put alternatively, $\rho_\Delta(T) := \mathbf{Cov}(Y(I), Y(J))$ where I and J are intervals of length Δ, and where the *centers* of I and J are distant T units of time apart.

Since $T \geqslant \Delta$ the intervals I and J are disjoint and hence Eq. (22) implies that

$$\rho_\Delta(T) = \int_T^{T+\Delta} \int_0^\Delta C(t-s)dt\ ds. \tag{24}$$

An immediate consequence of (24) is that if the auto-covariance $C(\cdot)$ is (asymptotically) monotone decreasing then $C(T+\Delta) \leqslant C(t-s) \leqslant C(T-\Delta)$, and hence the auto-correlation $\rho_\Delta(\cdot)$ retains the (asymptotic) shape of $C(\cdot)$:

$$\Delta^2 C(T+\Delta) \leqslant \rho_\Delta(T) \leqslant \Delta^2 C(T-\Delta). \tag{25}$$

A stochastic process is said to have short-range/long-range correlation if the tail integrals of its auto-correlation functions converge/diverge. It is self-evident from Eq. (25) that the integral $\int_{t_0}^\infty C(t)dt$ converges/diverges if and only if the integral $\int_{t_0}^\infty \rho_\Delta(t)dt$ converges/diverges. Hence, we can conclude that:

The pulsation process Y has short/long-range correlation
if and only if
the underlying rate process Λ has short/long-range correlation.

If, in addition to the (asymptotic) monotonicity, we also have

$$\lim_{T\to\infty} \frac{C(T+\delta)}{C(T)} = 1 \tag{26}$$

(for all real δ) then:

$$\rho_\Delta(T) \underset{T\to\infty}{\sim} \Delta^2 C(T). \tag{27}$$

That is, the auto-correlation function $\rho_\Delta(\cdot)$ is asymptotically identical (up to the multiplicative factor Δ^2) to the auto-covariance function $C(\cdot)$. Examples of asymptotically monotone decreasing auto-covariance functions $C(\cdot)$ satisfying condition (26) include:

1. sub-exponential decay: $C(T) \sim c\ \exp\{-bT^\beta\}$ ($c, b > 0$ and $0 < \beta < 1$),
2. power-law decay: $C(T) \sim bT^{-\beta}$ ($b, \beta > 0$),
3. logarithmic decay: $C(T) \sim b(\ln\{T\})^{-\beta}$ ($b, \beta > 0$).

Last, we note that taking the interval length Δ to zero and normalizing appropriately, Eq. (24) yields:

$$\lim_{\Delta\to 0} \frac{\rho_\Delta(T)}{\Delta^2} = C(T). \tag{28}$$

This result, which is analogous to Eq. (27) above, holds for all $T > 0$ and for all auto-covariance functions $C(\cdot)$.

3.3. Power Spectrum

An explicit correspondence between the auto-covariance function $C(\cdot)$ of the underlying rate process Λ, and the auto-correlation function $\rho_\Delta(\cdot)$ of the pulsation process Y, is attainable via their *power spectra*. Let $S(\omega)$ and $S_\Delta(\omega)$ (ω real) denote, respectively, the *spectral densities* of the auto-covariance and auto-correlation functions $C(\cdot)$ and $\rho_\Delta(\cdot)$. Namely:

$$\begin{aligned} C(T) &= \textstyle\int_{-\infty}^{\infty} \exp\{i\omega T\} S(\omega) d\omega, \\ \rho_\Delta(T) &= \textstyle\int_{-\infty}^{\infty} \exp\{i\omega T\} S_\Delta(\omega) d\omega. \end{aligned} \tag{29}$$

Then, the connection between the spectral densities $S(\cdot)$ and $S_\Delta(\cdot)$ is given by the following, simple, multiplicative relationship:

$$S_\Delta(\omega) = 2\frac{1-\cos(\Delta\omega)}{\omega^2} \cdot S(\omega). \tag{30}$$

The proof of Eq. (30) is given in the appendix (see Section A.2).

In particular, Eq. (30) implies that:

$$S_\Delta(\omega) \underset{\omega\to 0}{\sim} \Delta^2 S(\omega). \tag{31}$$

Equation (31) asserts that for small frequencies ($\omega \to 0$) the power spectrum $S_\Delta(\cdot)$ is identical (up to the multiplicative factor Δ^2) to the power spectrum $S(\cdot)$. In other words, Eq. (31) is the 'spectral counterpart' – in Fourier space – of Eq. (27). Furthermore Eq. (31) implies that the pulsation process Y is a '$1/f$ noise' with a given exponent, if and only if the underlying rate process Λ is a '$1/f$ noise' with the same exponent (for the notion of '$1/f$ noise' we refer the readers to ref. 34 and references therein).

Last, we note that when taking the interval length Δ to zero (and normalizing appropriately) we obtain the 'spectral counterpart' of Eq. (28):

$$\lim_{\Delta\to 0} \frac{S_\Delta(\omega)}{\Delta^2} = S(\omega). \tag{32}$$

4. STATIONARY RATE PROCESSES: EMISSION STRUCTURE

In the previous section, we studied the inter-dependencies of pulsation processes excited by general stationary rate processes. We did so by analyzing auto-correlations and power spectra – that is, by conducting a (first and) second order statistical analysis. However, first and second order statistical analysis sheds no light on the *emission structure* of the pulsation process Y. To that end we need to explore the behavior of the *emission statistics* introduced in Section 2: the cumulative pulsation; the time-to-pulsation; the inter-pulsation period; and, the pulsation multiplicity. Let us redefine these emission statistics for the general case where the underlying rate Λ is an arbitrary stochastic process:

- **Cumulative Pulsation** The cumulative number of pulsations during the time interval $(s, s+t)$: $Y(s+t) - Y(s)$.

- **Time-to-Pulsation** The time elapsing from time s till the first pulsation epoch occurring after it: $\tau(s) = \inf\{t \geqslant 0 | Y(s+t) - Y(s) > 0\}$. The probability distribution of $\tau(s)$ is given by:

$$\mathbf{P}(\tau(s) > t) = \mathbf{P}(Y(s+t) - Y(s) = 0) = \mathbf{E}[z^{Y(s+t)-Y(s)}]\Big|_{z=0}. \tag{33}$$

- **Inter-Pulsation Period** The time elapsing from time s till the first pulsation epoch occurring after it, *given* that a pulsation event occurred at time s. The inter-pulsation period $\tau_{\text{ip}}(s)$ is defined as the limit, *in law*, as $\delta \to 0$, of $\tau(s+\delta)$ *conditioned* on the event $\{Y(s+\delta) - Y(s) > 0\}$. The probability distribution of $\tau_{\text{ip}}(s)$ is hence given by:

$$\mathbf{P}\left(\tau_{\text{ip}}(s) > t\right) = \lim_{\delta\to 0} \mathbf{P}\left(\tau(s+\delta) > t \,|\, Y(s+\delta) - Y(s) > 0\right).$$

Using elementary calculations involving conditional probability we obtain that

$$\mathbf{P}\left(\tau_{\text{ip}}(s) > t\right) = \lim_{\delta\to 0} \frac{\mathbf{P}\left(\tau(s+\delta) > t\right) - \mathbf{P}\left(\tau(s) > t+\delta\right)}{1 - \mathbf{P}\left(\tau(s) > \delta\right)}. \tag{34}$$

- **Pulsation Multiplicity** The number of pulsations emitted simultaneously at time s, *given* that a pulsation event occurred at time s. The pulsation multiplicity $M(s)$ is defined as the limit, *in law*, as $\delta \to 0$, of $Y(s+\delta) - Y(s)$ *conditioned* on the event $\{Y(s+\delta) - Y(s) > 0\}$. The PGF of $M(s)$ is hence given by:

$$\mathbf{E}[z^{M(s)}] = \lim_{\delta\to 0} \mathbf{E}[z^{Y(s+\delta)-Y(s)} \,|\, Y(s+\delta) - Y(s) > 0].$$

Using Lemma 1 (see section A.1 of the appendix) and Eq. (33), we obtain that

$$\mathbf{E}[z^{M(s)}] = \lim_{\delta\to 0} \frac{\mathbf{E}[z^{Y(s+\delta)-Y(s)}] - \mathbf{P}\left(\tau(s) > \delta\right)}{1 - \mathbf{P}\left(\tau(s) > \delta\right)}. \tag{35}$$

When the underlying rate process is stationary – which is the case we aim to explore – then all four emission statistics defined above are *shift invariant*. That is, they are independent of the variable s. Therefore, we shall henceforth use the shorthand notation $Y(t)$, τ, τ_{ip}, and M to denote, respectively, the four emission statistics defined above.

4.1. Emission Statistics

Consider the case where the underlying rate $\Lambda = (\Lambda(t))_{t\geqslant 0}$ is an arbitrary stationary process, and set ($t, \theta \geqslant 0$):

$$\mathcal{H}(t;\theta) = -\ln \mathbf{E}\left[\exp\left\{-\theta \int_0^t \Lambda(u)du\right\}\right]. \tag{36}$$

Put equivalently, the function $\mathcal{H}(t;\theta)$ is implicitly defined by

$$\mathbf{E}\left[\exp\left\{-\theta\int_0^t \Lambda(u)du\right\}\right]=\exp\{-\mathcal{H}(t;\theta)\}.$$

Note that: **(i)** for all $t>0$ the function $\mathcal{H}(t;\cdot)$ begins at the origin, and is monotone increasing and concave; and, **(ii)** for all $\theta>0$ the function $\mathcal{H}(\cdot;\theta)$ begins at the origin and is monotone increasing.

The function $\mathcal{H}(t;\theta)$ defined above fully characterizes the emission statistics of the pulsation process Y, as we are now about to show.

The *cumulative pulsation* during a time interval of length t is governed by the PGF

$$\mathbf{E}[z^{Y(t)}]=\exp\{-\mathcal{H}(t;1-z)\}. \tag{37}$$

The proof of Eq. (37) is given in the appendix (see section A.3 of the appendix). An immediate consequence of Eq. (37) are the following formulae for the mean and variance of the cumulative pulsation (which are obtained by straightforward differentiation of Eq. (37) with respect to the variable z):

$$\mathbf{E}[Y(t)]=\frac{\partial\mathcal{H}}{\partial\theta}(t;0) \tag{38}$$

and

$$\mathbf{Var}(Y(t))=\frac{\partial\mathcal{H}}{\partial\theta}(t;0)-\frac{\partial^2\mathcal{H}}{\partial\theta^2}(t;0). \tag{39}$$

Note that Eqs. (38) and (39) are alternative representations of Eqs. (20) and (21), derived in the previous section (using first and second-order analysis).

Yet another immediate consequence of Eq. (37) (using Eq. (33)) is the distribution of the *time-to-pulsation* ($t\geqslant 0$):

$$\mathbf{P}(\tau>t)=\exp\{-\mathcal{H}(t;1)\}. \tag{40}$$

Furthermore, Eqs. (37) and (40) enable us to compute the right-hand-side limits in Eqs. (34) and (35) which yield, respectively: **(i)** the probability distribution of the *inter-pulsation period*

$$\mathbf{P}\left(\tau_{\mathrm{ip}}>t\right)=\exp\{-\mathcal{H}(t;1)\}\,\frac{\frac{\partial\mathcal{H}}{\partial t}(t;1)}{\frac{\partial\mathcal{H}}{\partial t}(0;1)}. \tag{41}$$

and, **(ii)** the PGF of the *pulsation multiplicity*

$$\mathbf{E}[z^M] = 1 - \lim_{\delta\to 0}\frac{\mathcal{H}(\delta; 1-z)}{\mathcal{H}(\delta; 1)} = 1 - \frac{\frac{\partial\mathcal{H}}{\partial t}(0; 1-z)}{\frac{\partial\mathcal{H}}{\partial t}(0; 1)}. \tag{42}$$

The proof of Eqs. (41) and (42) is given in the appendix (see section A.3).

4.2. The Inter-Pulsation Periods

Are the inter-pulsation periods governed by a *proper* probability law? The answer to this question is determined by the mean, $\lambda = \mathbf{E}[\Lambda(t)]$, of the underlying rate process Λ. We explain.

First, note that differentiating Eq. (36) gives $(\partial\mathcal{H}/\partial t)(0;\theta) = \lambda\theta$ $(\theta \geqslant 0)$, and hence:

$$\frac{\partial\mathcal{H}}{\partial t}(0; 1) = \lambda. \tag{43}$$

The finiteness of $(\partial\mathcal{H}/\partial t)(0; 1)$ determines, in turn, the *properness* of the distribution of the inter-pulsation period:

- If the mean of the underlying rate process Λ is *infinite* $(\lambda = \infty)$ then the substitution of Eq. (43) into Eq. (41) implies that $\mathbf{P}\left(\tau_{\text{ip}} > t\right) = 0$ for all $t > 0$ – rendering the inter-pulsation period *degenerate*:

$$\tau_{\text{ip}} \equiv 0. \tag{44}$$

The meaning of Eq. (44) is that the notion of the "time period between consecutive pulsation epochs" is indeterminate in the case of underlying rates with infinite mean.

- On the other hand, if the mean of the underlying rate process Λ is *finite* $(\lambda < \infty)$ then Eq. (41) implies that the distribution of the inter-pulsation period τ_{ip} is *proper*, and that its mean is given by:

$$\mathbf{E}[\tau_{\text{ip}}] = \int_0^\infty \mathbf{P}\left(\tau_{\text{ip}} > t\right) dt = \frac{1}{\lambda}. \tag{45}$$

Furthermore, substituting $(\partial\mathcal{H}/\partial t)(0; 1) = \lambda$ and $(\partial\mathcal{H}/\partial t)(0; 1-z) = \lambda(1-z)$ into Eq. (42) yields $\mathbf{E}[z^M] = z$ which, in turn, implies that

$$M \equiv 1. \tag{46}$$

That is, in the finite-mean case the process Y emits *single* pulses, and the distribution of the inter-pulsation periods is proper with mean $1/\lambda$.

Let us now turn to study, in further detail, the finite-mean ($\lambda < \infty$) and infinite-mean ($\lambda = \infty$) cases.

4.3. The Finite-Mean Case

The *hazard rate function* $h_\xi(t)$ ($t \geqslant 0$) of a non-negative random variable ξ is defined as the local rate of occurrence of ξ at time t, *given* that it did not occur up to time t.[35] Namely:

$$h_\xi(t) = \lim_{\delta \to 0} \frac{1}{\delta} \mathbf{P}(t < \xi \leqslant t + \delta | \xi > t) = \frac{F'_\xi(t)}{1 - F_\xi(t)},$$

where $F_\xi(t) = \mathbf{P}(\xi \leqslant t)$ stands for the cumulative distribution function of ξ. The correspondence between distribution functions and hazard rate functions is one-to-one. Indeed, the distribution of ξ is uniquely determined by its hazard rate function via

$$\mathbf{P}(\xi > t) = \exp\left\{ - \int_0^t h_\xi(u) du \right\}. \tag{47}$$

The concept of *hazard rate* plays an important role in various fields of applied probability, and is central in reliability theory. For a comprehensive treatment of hazard rate functions we refer the reader to ref. 36.

From Eq. (40) we immediately obtain that the hazard rate function of the time-to-pulsation τ is given by:

$$h(t) = \frac{\partial \mathcal{H}}{\partial t}(t; 1). \tag{48}$$

Note that Eq. (43) implies that $h(0) = \lambda$.

In the finite-mean case ($\lambda < \infty$) the inter-pulsation period is well defined, and it's distribution is given by Eq. (41). Substituting τ's hazard rate function (48) into Eq. (41), and using the representation (47) (for the random variable τ), yields:

$$\mathbf{P}\left(\tau_{\rm ip} > t\right) = \exp\left\{ - \int_0^t h(u) du \right\} \tfrac{h(t)}{h(0)} = \exp\left\{ - \int_0^t \left(h(u) - \tfrac{h'(u)}{h(u)} \right) du \right\}.$$

Hence, we have obtained an explicit correspondence between the hazard rate function of the inter-pulsation period – denote it by $h_{\rm ip}(t)$ – and the hazard rate function of the time-to-pulsation τ:

$$h_{\mathrm{ip}}(t) = h(t) - \frac{h'(t)}{h(t)}. \tag{49}$$

In terms of the function $\mathcal{H}(t;\theta)$ the hazard rate of the inter-pulsation periods is given by

$$h_{\mathrm{ip}}(t) = \frac{\partial \mathcal{H}}{\partial t}(t;1) - \frac{\frac{\partial^2 \mathcal{H}}{\partial t^2}(t;1)}{\frac{\partial \mathcal{H}}{\partial t}(t;1)}. \tag{50}$$

It is interesting to compare a Poissonian pulsation process with (deterministic) rate λ to the pulsation process Y excited by a random stationary rate process Λ with mean λ. In both cases the inter-pulsation period are random with mean length $1/\lambda$. In the Poissonian case the inter-pulsation periods are exponentially-distributed – rendering the process *Markovian* and *Lévy*. When we replace the deterministic rate λ with a random stationary rate process Λ we induce a flow of information across the time axis. This flow of information breaks both the Markov and Lévy properties, and results in a change of the distribution of the inter-pulsation periods. The distribution of the new, non-Markovian, inter-pulsation periods is governed by the hazard rate function $h_{\mathrm{ip}}(t)$ given above.

The information flow induced by the rate process Λ also *reduces the entropy* of the inter-pulsation periods, as we shall now explain. Consider the set of all probability distributions on the non-negative half-line, having mean $1/\lambda$. Amongst these distributions, the one with maximal entropy is the exponential distribution with rate λ (see, for example, ref. 37). Furthermore, the exponential distribution is the unique maxima – all other distribution have strictly lesser entropy. Hence, the change of the inter-pulsation distribution caused by the underlying stationary rate process Λ also reduces the entropy: from the maximal entropy of the exponential inter-pulsation, to the lesser entropy of τ_{ip}.

4.4. The Infinite-Mean Case

As we have seen in section 4.2, if the underlying stationary rate process Λ has infinite mean ($\lambda = \infty$) then the consecutive pulsation epochs of the pulsation process Y fail to be temporally separated. In other words, contrary to the finite-mean case – the temporal structure of the Poisson process N is *not* preserved by the pulsation process Y.

Is there more to be said about the infinite-mean case? The answer is affirmative – provided that the 'type of the infiniteness' of the mean of Λ is specified in finer detail, and provided that Λ satisfies a certain regularity requirement. Namely, we assume that:

(i) The stationary distribution of Λ is heavy-tailed of order α $(0 < \alpha < 1)$:

$$\mathbf{P}(\Lambda(t) > x) \underset{x\to\infty}{\sim} \frac{1}{\Gamma(1-\alpha)} \frac{l(x)}{x^{\alpha}}, \tag{51}$$

where $l(\cdot)$ is a slowly varying function at infinity (see, for example, ref. 33).

(ii) For all $\theta > 0$, the following functions (in t) are asymptotically equivalent:

$$1 - \mathbf{E}\left[\exp\left\{-\theta \int_0^t \Lambda(u)du\right\}\right] \underset{t\to 0}{\sim} 1 - \mathbf{E}\left[\exp\{-\theta\Lambda(0)t\}\right]. \tag{52}$$

The meaning of assumption (52) is that the random variable $\int_0^t \Lambda(u)du$ is close, in law, to the random variable $\Lambda(0)t$, as $t \to 0$.

If these assumptions hold then the time-to-pulsation displays an *anomalous* short-time behavior:

$$\mathbf{P}(\tau \leqslant t) \underset{t\to 0}{\sim} t^{\alpha} l(1/t). \tag{53}$$

The proof of Eq. (53) is given in the appendix (see section A.3). This behavior stands in contrast to the *linear* short-time behavior taking place in the finite-mean case:

$$\mathbf{P}(\tau \leqslant t) \underset{t\to 0}{\sim} \lambda t$$

(this equation is an immediate consequence of Eqs. (40) and (43)).

Furthermore, if Eqs. (51) and (52) hold then the pulsation multiplicity of the pulsation process Y is governed by the PGF

$$\mathbf{E}[z^M] = 1 - (1-z)^{\alpha}. \tag{54}$$

The proof of Eq. (54) is given in the appendix (see section A.3).

We have already encountered the PGF of Eq. (54) in subsection 2.5 where it appeared as the PGF of the pulsation multiplicity of a Poisson process subordinated by an α-*selfsimilar* Lévy motion (recall Equation (15)). It is rather surprising to see that the law of the pulsation multiplicity turns out to be *universal* – as it depends solely on the 'order of the heaviness' $0<\alpha<1$ of the stationary distribution of the underlying rate process Λ. The structure of Λ, the dynamics of Λ, the slowly varying function $l(\cdot)$ – are all irrelevant factors when it comes to the determination of the pulsation multiplicity. The one factor alone which 'calls the shots' is the order of the heaviness α.

5. ORNSTEIN–UHLENBECK RATES

In this section, we explore the case where the stochastic rate process $\Lambda=(\Lambda(t))_{t\geqslant 0}$ is a Lévy-driven Ornstein–Uhlenbeck process.[27,38] Namely, Λ is given by the stochastic dynamics

$$\Lambda(dt)=-\kappa\Lambda(t)dt+X(dt), \tag{55}$$

where κ is a positive constant, and where $X=(X(t))_{t\geqslant 0}$ is a one-sided Lévy motion with Lévy characteristic $\phi(\omega)$ ($\omega\geqslant 0$). We emphasize that the Ornstein–Uhlenbeck rate process Λ is *Markovian*.

The stochastic differential equation (55) admits the explicit solution

$$\Lambda(t)=\exp\{-\kappa t\}\Lambda(0)+\int_0^t \exp\{-\kappa(t-s)\}X(ds), \tag{56}$$

where the initial value $\Lambda(0)$ and the Lévy driver X are *independent*. Moreover, if the Ornstein–Uhlenbeck process Λ is *stationary* then the Laplace transform of $\Lambda(t)$ is given by (see, for example, ref. 27) ($t,\theta\geqslant 0$):

$$\mathbf{E}[\exp\{-\theta\Lambda(t)\}]=\exp\left\{-\frac{1}{\kappa}\int_0^\theta \frac{\phi(u)}{u}du\right\}. \tag{57}$$

Based on Eqs. (56) and (57) we obtain that the function $\mathcal{H}(t;\theta)$ (defined in Eq. (36)) for the stationary Ornstein–Uhlenbeck rate process Λ governed by the dynamics (55), is given by ($t,\theta\geqslant 0$):

$$\mathcal{H}(t;\theta)=\int_0^t \frac{\phi\left(\frac{\theta}{\kappa}(1-\exp\{-\kappa u\})\right)}{1-\exp\{-\kappa u\}}du. \tag{58}$$

The proof of Eq. (58) is given in the appendix (see section A.4).

'Armed' with Eq. (58), we are now ready to apply the general results obtained in Section 4 to the class of pulsation processes excited by stationary Lévy-driven Ornstein–Uhlenbeck rates. However, this is merely a special case amongst a far larger class of pulsation processes excited by Lévy-driven Moving-Average rates. Hence, rather than studying the particular Ornstein–Uhlenbeck example, we turn now to explore its 'mother class' of Moving-Average rates.

6. MOVING-AVERAGE RATES

If we allow the rate processes to be defined on the entire real line (rather than on the non-negative half-line $t \geqslant 0$), then the stationary Ornstein–Uhlenbeck rate process of the previous section admits the representation

$$\Lambda(t) = \int_{-\infty}^{t} \exp\{-\kappa(t-s)\} X(ds)$$

(the one-sided Lévy driver X is taken now to be defined on the entire real line). That is, Λ is a *Moving-Average* of the Lévy driver X, with an exponential convolution kernel.

In general, we can consider Moving-Average rate processes $\Lambda = (\Lambda(t))_{-\infty<t<\infty}$ of the form:

$$\Lambda(t) = \int_{-\infty}^{t} f(t-s) X(ds), \tag{59}$$

where $f(t)$ $(t \geqslant 0)$ is an arbitrary non-negative *impulse-response* function, and where $X = (X(t))_{-\infty<t<\infty}$ is a one-sided Lévy motion (defined on the real line) with Lévy characteristic $\phi(\omega)$ $(\omega \geqslant 0)$.

Let us denote by $F(t)$ $(t \geqslant 0)$ the primitive of the impulse-response function. Namely; $F(t) = \int_0^t f(u)du$. With this notation at hand, we obtain that $(t \geqslant 0)$:

$$\int_0^t \Lambda(u)du = \int_{-\infty}^{0} (F(t-u) - F(-u))\, X(du) + \int_0^t F(t-u) X(du), \tag{60}$$

which, in turn, implies that $(t, \theta \geqslant 0)$:

$$\mathcal{H}(t;\theta) = \int_0^\infty \phi\left(\theta\left(F(u+t) - F(u)\right)\right) du + \int_0^t \phi\left(\theta F(u)\right) du. \tag{61}$$

Closing the Circle: Back to Subordination

Equation (60) enables us to 'close the circle' initiated in Section 2. Let us first introduce the process $X_F = (X_F(t))_{t \geqslant 0}$ defined by

$$X_F(t) = \int_{-\infty}^{\infty} (F(t-u) - F(-u))\, X(du) \ , \tag{62}$$

where we set $F(t) \equiv 0$ for $t \leqslant 0$. Since $X_F(t) = \int_0^t \Lambda(u)du$ we obtain the following representation of the pulsation process Y:

$$Y(t) = N(X_F(t)). \tag{63}$$

That is, the pulsation process $Y = (Y(t))_{t \geqslant 0}$ is given by the *subordination* of the Poisson process N to the process X_F.

The subordinating process X_F, defined in Eq. (62), is a linear transformation $X \mapsto X_F$ of the driving Lévy motion X. This transformation preserves the input's stationary-increments property, but does not preserve the independent-increments property: the increments of the output process X_F are stationary, but need *not* be independent.

Equations (62) and (63) can be henceforth used as our 'starting point' (rather than the original Moving-Average representation (59)). We need, of course, to require the admissability of $F(\cdot)$: **(i)** it should be non-negative valued and non-decreasing; and, **(ii)** the right-hand-side of Eq. (61) should be finite for all $t, \theta \geqslant 0$.

If, for example, we take $F(t) \equiv 1$ then $X_F \equiv X$ and we hence return to the Lévy subordination case studied in Section 2. On the other hand, if we take $F(t) = \kappa^{-1}(1 - \exp\{-\kappa t\})$ then we obtain the case of Ornstein–Uhlenbeck excitation presented in Section 5.

Last, we note that the structure of X_F – a stochastic integral process with respect to an underlying driving Lévy motion X – is analogous to structure of *fractional Brownian motion* and *fractional Lévy motion* (see, for example, ref. 32).

6.1. Finite-Mean Drivers

In this section, we consider the case of finite-mean Lévy drivers. That is, the case of a Lévy driver X satisfying

$$\mathbf{E}[X(1)] = \phi'(0) = \mu < \infty.$$

In this case, a necessary and sufficient condition for the convergence of the integral on right-hand-side of Eq. (61) is $\int_0^\infty (F(u+t) - F(u))\, du < \infty$

which, in turn, holds if and only if $F(\infty) := \lim_{u\to\infty} F(u) < \infty$. Hence, the admissibility condition for the well-posedness of the subordinating process X_F is that $F(\cdot)$ is non-negative valued, non-decreasing, and bounded. In other words, $F(\cdot)$ should be – up to a multiplicative constant – a cumulative distribution function of a probability distribution on the non-negative half-line.

Using the general results of Section 4.1 we obtain that:

- The distribution of the time-to-pulsation is $\mathbf{P}(\tau > t) = \exp\{-\mathcal{H}(t;1)\}$. The function $\mathcal{H}(t;1)$ – which is monotone increasing – has the following asymptotes at the origin and at infinity:

$$\mathcal{H}(t;1) \sim \begin{cases} a_0 \cdot t & \text{as } t \to 0, \\ a_\infty \cdot t + b_\infty & \text{as } t \to \infty, \end{cases} \tag{64}$$

where $a_0 = \phi(F(0)) + \mu(F(\infty) - F(0))$; $a_\infty = \phi(F(\infty))$; and where $b_\infty = \int_0^\infty (\phi(F(u)) + \phi(F(\infty) - F(u)) - \phi(F(\infty)))\, du$.

- The inter-pulsation periods are well-defined if and only if $(\partial\mathcal{H}/\partial t)(0;1) < \infty$ – which is indeed satisfied since:

$$\frac{\partial\mathcal{H}}{\partial t}(0;1) = \phi(F(0)) + \mu(F(\infty) - F(0)) < \infty.$$

- The PGF of the pulsation multiplicity is given by:

$$\mathbf{E}[z^M] = 1 - \frac{\frac{\partial\mathcal{H}}{\partial t}(0;1-z)}{\frac{\partial\mathcal{H}}{\partial t}(0;1)} = \frac{\phi(F(0)) - \phi(F(0)(1-z)) + \mu(F(\infty) - F(0))z}{\phi(F(0)) + \mu(F(\infty) - F(0))}. \tag{65}$$

Furthermore, the process Y emits single pulses if and only if $\mathbf{E}[z^M] = z$ – which holds, due to Eq. (65), if and only if $F(0) = 0$.

6.2. Heavy-Tailed Drivers

In this section, we consider the case of heavy-tailed Lévy drivers. That is, the case of a Lévy driver X satisfying

$$\mathbf{P}(X(1) > x) \underset{x\to\infty}{\sim} \frac{1}{\Gamma(1-\alpha)} \frac{l(x)}{x^\alpha},$$

where $0 < \alpha < 1$ is the 'tail order' and where $l(\cdot)$ is a slowly varying function at infinity.[33] Put equivalently[3], we consider the case where the driver's Lévy characteristic satisfies

$$\phi(\omega) \underset{\omega\to 0}{\sim} \omega^{\alpha} l(1/\omega). \tag{66}$$

We introduce the notation

$$\mathcal{F}_{\alpha}(t) = \int_0^{\infty} (F(u+t) - F(u))^{\alpha}\, du + \int_0^t F(u)^{\alpha} du. \tag{67}$$

The finiteness of $\mathcal{F}_{\alpha}(t)$ is a necessary and sufficient condition for the well-posedness of the subordinating process X_F. Note that the function $\mathcal{F}_{\alpha}(t)$ equals the function $\mathcal{H}(t; 1)$ – the logarithm of the distribution of the time-to-pulsation – in the case where the Lévy driver X is α-selfsimilar: $\phi(\omega) = \omega^{\alpha}$.

Using the general results of Section 4.1 we obtain that:

- The increments of the pulsation process Y are heavy tailed:

$$\mathbf{P}(Y(t) > y) \underset{y\to\infty}{\sim} \frac{\mathcal{F}_{\alpha}(t)}{\Gamma(1-\alpha)} \cdot \frac{l(1/y)}{y^{\alpha}}. \tag{68}$$

The proof of Eq. (68) is given in the appendix (see section A.5).

- The short-time behavior of the time-to-pulsation is *anomalous*:

$$\mathbf{P}(\tau \leqslant t) = 1 - \exp\{-\mathcal{H}(t; 1)\} \underset{t\to 0}{\sim} c_{\alpha} \cdot \phi(t),$$

where $c_{\alpha} = \int_0^{\infty} f(u)^{\alpha} du$. This behavior is different from the *linear* short-time behavior of the time-to-pulsation in the case of finite-mean Lévy drivers: $\mathbf{P}(\tau \leqslant t) \sim \mu t$ as $t \to 0$. However, the asymptotic behavior of the function $\mathcal{H}(t; 1)$ at infinity is the same as in the finite-mean case (see Eq. (64)).

- The inter-pulsation periods are well-defined if and only if $(\partial\mathcal{H}/\partial t)(0; 1) < \infty$ - which is *never* satisfied since:

$$\frac{\partial \mathcal{H}}{\partial t}(0; 1) = \lim_{t\to 0} \mathcal{F}'_{\alpha}(t) = \infty.$$

[3]Using Karamata's Tauberian theorem for random variables (see, for example, corollary 8.1.7 in ref 33).

- The PGF of the pulsation multiplicity is given by:

$$\mathbf{E}[z^M] = 1 - \lim_{\delta \to 0} \frac{\mathcal{H}(\delta; 1-z)}{\mathcal{H}(\delta; 1)} = 1 - (1-z)^\alpha$$

(to see this, note that for all $\theta > 0$ we have $\mathcal{H}(t;\theta) \sim c_\alpha \theta^\alpha \cdot \phi(t)$ as $t \to 0$). As discussed in Section 4.4, the distribution of the pulsation multiplicity is *universal* – it depends neither on the fine structure of the Lévy driver (determined by the slowly varying function $l(\cdot)$), nor on the averaging structure (determined by the function $F(\cdot)$). Rather, it depends *solely* on the order $0 < \alpha < 1$ of the Lévy driver's tails.

6.3. Auto-Correlation

In this subsection, we consider the case of Lévy drivers with finite variance. That is, the case of a Lévy driver X satisfying

$$\mathbf{Var}(X(1)) = -\phi''(0) = \sigma^2 < \infty \ ,$$

and study the inter-dependence of the pulsation process Y. Recall our definition for the auto-correlation function of Y ($T \geqslant \Delta$):

$$\rho_\Delta(T) := \mathbf{Cov}\,(Y(T+\Delta) - Y(T), Y(\Delta)) \ .$$

Using the subordination representation (63) we obtain that

$$\rho_\Delta(T) = \sigma^2 \int_0^\infty [F(u+T) - F(u+T-\Delta)][F(u) - F(u-\Delta)]du \ . \quad (69)$$

The proof of Eq. (69) is given in the appendix (see section A.5).

If the impulse-response function satisfies $\lim_{T\to\infty} f(T+\delta)/f(T) = 1$ (the convergence holding uniformly on compact sets), then the auto-correlation function $\rho_\Delta(\cdot)$ 'inherits' the asymptotic behavior of $f(\cdot)$:

$$\rho_\Delta(T) \underset{T\to\infty}{\sim} \Delta^2\sigma^2(F(\infty) - F(0)) \cdot f(T) \ .$$

Last, we note that taking the interval length Δ to zero and normalizing appropriately, Eq. (69) yields

$$\lim_{\Delta \to 0} \frac{\rho_\Delta(T)}{\Delta^2} = \sigma^2 \int_0^\infty f(u+T) f(u) du \ . \quad (70)$$

The right hand side of Eq. (70) is the auto-covariance function $C(T)$ of the Moving-Average rate process Λ given in Eq. (59). Hence, Eq. (70) is in agreement with the general result (28) obtained in Section 3.2.

Example: Ornstein–Uhlenbeck Rates
As mentioned above, in the Ornstein–Uhlenbeck case the impulse-response function is $f(t) = \exp\{-\kappa t\}$, and hence $F(t) = \kappa^{-1}(1 - \exp\{-\kappa t\})$ and $C(T) = (\sigma^2/2\kappa)\exp\{-\kappa|T|\}$ (the auto-covariance $C(T)$ is straightforwardly deduced from the Moving-Average representation (59)). Now, calculating the right-hand-side of either Eq. (69) or Eq. (24) yields the following explicit formula for the auto-correlation function:

$$\rho_\Delta(T) = \frac{\sigma^2}{\kappa^3}(\cosh(\kappa\Delta) - 1) \cdot \exp\{-\kappa T\} .$$

7. CONCLUSIONS

The Poisson Process and Brownian motion are the best-known examples of the family of Lévy processes. These two examples are, in essence, 'orthogonal'. The Poisson process is discrete in nature, and serves as the predominant 'model-of-choice' to describe random pulsations. Brownian motion, on the other hand, is continuous in nature, and serves as the predominant 'model-of-choice' to describe diffusive propagation. However, both examples share the fundamental 'Lévy property' – their increments are stationary and independent.

It is well known that anomalous diffusion is attainable from regular (Brownian) diffusion via temporal subordination. Specifically, subjecting Brownian motion to a random time flow can yield both sub-diffusive and super-diffusive behaviors. Thus, anomalous diffusive motion is obtained not by distorting or changing the underlying transport mechanism, but by temporal subordination to a 'randomized operational time' (as referred to by Feller[(23)]).

Motivated by the following facts:

- the fundamental 'Lévy property' shared by both the Poisson process and Brownian motion,
- the ability to derive anomalous diffusion out of regular diffusion via temporal subordination, and,
- the blinking phenomena observed in a wide range of physical systems;

we ponder: could Poissonian-based anomalous pulsation models – analogous and counterpart to the Brownian-based anomalous diffusion models – be successfully constructed? And, if yes, what type of behaviors would such anomalous pulsation models display? This manuscript is devoted to the exploration of these questions. To do so, we considered a standard Poisson process 'stimulated' by a random stationary rate (rather than by a constant deterministic rate).

Based on the analogy to anomalous diffusion, we began with the study of the case where the random stationary rate process is a one-sided Lévy noise (i.e., the derivative one-sided Lévy motion). This gave rise to the phenomena of *multiple pulsations*: the resulting pulsation process no longer fires 'semi-automatically' – one pulse at a time; rather, it fires 'automatically' – in bursts of random size. We termed the random burst-size "pulsation multiplicity", and studied its distribution and its dependence on the driving Lévy noise.

We then turned to study the general case of pulsation processes stimulated by arbitrary random stationary rate processes. First, we reviewed the standard approach of analyzing the first and second order statistics and correlations: mean; variance; auto-covariance; and, power-spectrum. Second, we studied the emission structure of the pulsation process, analyzing the following emission statistics:

- **Cumulative Pulsation** – the cumulative number of pulsations emitted during a time interval of a given length;
- **Time-to-Pulsation** – the time an external observer, having started his watch at given time epoch, will have to wait till encountering a pulsation event;
- **Inter-Pulsation Period** – the time elapsing between consecutive pulsation events; and,
- **Pulsation Multiplicity** – the number of pulsations emitted during a single pulsation event.

The analysis conducted pointed out a sharp and dramatic distinction between the cases of pulsation processes stimulated by finite-mean and infinite-mean stationary rates. In the first case – finite-mean rates – the resulting pulsation process is regular and has an emission structure topologically equivalent to the standard Poissonian one: pulsation events are temporally spaced, and emission are always single. In the case of infinite-mean rates, however, the situation is truly anomalous: (i) the local rates of pulsation are non-linear; (ii) pulsation epochs cluster and the notion of inter-pulsation periods is indeterminate; and, (iii) emissions are fired in bursts.

We concluded with employing the general theory developed to the case of pulsation processes stimulated by Lévy-driven Moving-Average rates. This class of stationary rate processes turns out to be a ‘mother model’ to many specific rates including: the Lévy-noise example we began with; Lévy-driven Ornstein–Uhlenbeck rates; and, ‘fractional-rates’ whose underlying structure and memory is identical to that of fractional Brownian and Lévy motions.

APPENDIX A

A.1. A Useful Lemma

Lemma 1. Let R be an integer-valued random variable with PGF $G(z)$ ($|z|\leqslant 1$). Then, the conditional PGF of R, conditioned on the event $\{R>0\}$, is given by:

$$\mathbf{E}[z^R|R>0]=\frac{G(z)-G(0)}{1-G(0)} \ .$$

Proof. Let I_0 denote the indicator of the event $\{R=0\}$ (hence $1-I_0$ is the indicator of the event $\{R>0\}$). Then:

$$\mathbf{E}[z^R|R>0]=\frac{\mathbf{E}[z^R(1-I_0)]}{\mathbf{E}[(1-I_0)]} \ . \tag{A1}$$

However,

$$\begin{aligned}\mathbf{E}[z^R(1-I_0)]&=\mathbf{E}[z^R]-\mathbf{E}[z^R I_0]\\&=\mathbf{E}[z^R]-\mathbf{P}(R=0)=G(z)-G(0) \ ,\end{aligned} \tag{A2}$$

and

$$\mathbf{E}[(1-I_0)]=1-\mathbf{P}(R=0)=1-G(0) \ . \tag{A3}$$

Substituting Eqs. (A2) and (A3) back into Eq. (A1) completes the proof. ∎

A.2. Proofs: Section 3

Equations (20)–(22): Mean, Variance, and Covariance

Proof. *Given* the sample path of the rate process Λ, the random variables $Y(I)$ and $Y(J)$ are Poisson-distributed, with parameters $\int_I \Lambda(t)dt$ and $\int_J \Lambda(s)ds$, respectively. Moreover, their conditional covariance (*given* Λ) equals the conditional variance (*given* Λ) of the 'intersection cumulative pulsation' $Y(I \cap J)$. Since both the mean and variance of a Poisson-distributed random variable are equal to its parameter, we obtain that:

$$\mathbf{E}[Y(I)|\Lambda] = \int_I \Lambda(t)dt; \quad \mathbf{E}[Y(J)|\Lambda] = \int_J \Lambda(s)ds$$

and,

$$\mathbf{Cov}(Y(I), Y(J)|\Lambda) = \mathbf{Var}(Y(I \cap J)|\Lambda) = \int_{I \cap J} \Lambda(t)dt.$$

Hence, using conditioning, we have:

$$\begin{aligned}\mathbf{E}[Y(I)] &= \mathbf{E}[\mathbf{E}[Y(I)|\Lambda]] \\ &= \mathbf{E}\left[\int_I \Lambda(t)dt\right] = \int_I \mathbf{E}[\Lambda(t)]dt = \lambda|I|\end{aligned}$$

proving Eq. (20); and,

$$\begin{aligned}&\mathbf{Cov}(Y(I), Y(J)) \\ &\quad = \mathbf{E}[\mathbf{Cov}(Y(I), Y(J)|\Lambda)] + \mathbf{Cov}(\mathbf{E}[Y(I)|\Lambda], \mathbf{E}[Y(J)|\Lambda]) \\ &\quad = \mathbf{E}\left[\int_{I \cap J} \Lambda(t)dt\right] + \mathbf{Cov}\left(\int_I \Lambda(t)dt, \int_J \Lambda(s)ds\right) \\ &\quad = \int_{I \cap J} \mathbf{E}[\Lambda(t)]dt + \int_I \int_J \mathbf{Cov}(\Lambda(t), \Lambda(s))\, dt\, ds \\ &\quad = \lambda|I \cap J| + \int_I \int_J C(t-s)dt\, ds\end{aligned}$$

proving Eq. (22).

Last, Eq. (21) follows immediately from Eq. (22) by taking $J = I$. ∎

Equation (30): Spectral Density

Proof. Using Eq. (24) for the auto-correlation function $\rho_\Delta(\cdot)$ and the spectral representation (29) of the auto-covariance function $C(\cdot)$, we have

$$\begin{aligned}\rho_\Delta(T) &= \int_T^{T+\Delta}\int_0^\Delta C(t-s)dt\ ds\\ &= \int_T^{T+\Delta}\int_0^\Delta \left(\int_{-\infty}^{\infty}\exp\{i\omega(t-s)\}S(\omega)d\omega\right)dt\ ds\\ &= \int_{-\infty}^{\infty}\left(\int_T^{T+\Delta}\exp\{i\omega t\}dt\right)\left(\int_0^\Delta \exp\{-i\omega s\}ds\right)S(\omega)d\omega\\ &= \int_{-\infty}^{\infty}\left(\frac{1-\exp\{i\omega\Delta\}}{-i\omega}\exp\{i\omega T\}\right)\left(\frac{1-\exp\{-i\omega\Delta\}}{i\omega}\right)S(\omega)d\omega\\ &= \int_{-\infty}^{\infty}\exp\{i\omega T\}\left(2\frac{1-\cos(\Delta\omega)}{\omega^2}S(\omega)\right)d\omega\ .\end{aligned}$$

This, in turn, implies that $2\omega^{-2}(1-\cos(\Delta\omega))S(\omega)$ is the spectral density of the auto-correlation function $\rho_\Delta(\cdot)$. ∎

A.3. Proofs: Section 4

Equation (37): The PGF of the Cumulative Pulsation

Proof. Using conditioning, the stationarity of the underlying rate process Λ, and the definition of the function $\mathcal{H}(t;\theta)$ (given in Eq. (36)), we have:

$$\begin{aligned}\mathbf{E}[z^{Y(s+t)-Y(s)}] &= \mathbf{E}\left[\mathbf{E}\left[z^{N\left(\int_0^{s+t}\Lambda(u)du\right)-N\left(\int_0^s\Lambda(u)du\right)}|\Lambda\right]\right]\\ &= \mathbf{E}\left[\exp\left\{-(1-z)\int_s^{s+t}\Lambda(u)du\right\}\right] = \mathbf{E}\left[\exp\left\{-(1-z)\int_0^t\Lambda(u)du\right\}\right]\\ &= \exp\{-\mathcal{H}(t;1-z)\}.\end{aligned}$$ ∎

Equation (41): The Inter-Pulsation Period

Proof. Using the distribution of the time-to-pulsation (Eq. (40)), we have

$$\begin{aligned}\frac{\mathbf{P}(\tau(s+\delta)>t)-\mathbf{P}(\tau(s)>t+\delta)}{1-\mathbf{P}(\tau(s)>\delta)} &= \frac{\exp\{-\mathcal{H}(t;1)\}-\exp\{-\mathcal{H}(t+\delta;1)\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}\\ &= \exp\{-\mathcal{H}(t;1)\}\frac{1-\exp\{-(\mathcal{H}(t+\delta;1)-\mathcal{H}(t;1))\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}\ .\end{aligned} \tag{A4}$$

On the other hand, using L'Hospital's rule we obtain that

$$\lim_{\delta\to 0}\frac{1-\exp\{-(\mathcal{H}(t+\delta;1)-\mathcal{H}(t;1))\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}=\lim_{\delta\to 0}\frac{\mathcal{H}(t+\delta;1)-\mathcal{H}(t;1)}{\mathcal{H}(\delta;1)}$$
$$=\lim_{\delta\to 0}\frac{(\mathcal{H}(t+\delta;1)-\mathcal{H}(t;1))/\delta}{(\mathcal{H}(\delta;1)-\mathcal{H}(0;1))/\delta}=\left(\frac{\partial\mathcal{H}}{\partial t}(t;1)\right)\Big/\left(\frac{\partial\mathcal{H}}{\partial t}(0;1)\right). \tag{A5}$$

Substituting Eq. (A4) into the right-hand-side of Eq. (34) (the distribution of the inter-pulsation period) and using the limit (A5) yields:

$$\mathbf{P}\left(\tau_{\mathrm{ip}}>t\right)=\exp\{-\mathcal{H}(t;1)\}\,\frac{\frac{\partial\mathcal{H}}{\partial t}(t;1)}{\frac{\partial\mathcal{H}}{\partial t}(0;1)}\ .\quad\blacksquare$$

Equation (42): The Pulsation Multiplicity

Proof. Using the PGF of the cumulative pulsation (given in Eq. (37)) and the distribution function of the time-to-pulsation (given in Eq. (40)), we have

$$\frac{\mathbf{E}[z^{Y(s+\delta)-Y(s)}]-\mathbf{P}(\tau(s)>\delta)}{1-\mathbf{P}(\tau(s)>\delta)}=\frac{\exp\{-\mathcal{H}(\delta;1-z)\}-\exp\{-\mathcal{H}(\delta;1)\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}$$
$$=1-\frac{1-\exp\{-\mathcal{H}(\delta;1-z)\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}\ . \tag{A6}$$

Substituting (A6) into the right-hand-side of Eq. uation (35) (the PGF of the pulsation multiplicity) and using L'Hospital's rule yields

$$\mathbf{E}[z^M]=1-\lim_{\delta\to 0}\frac{\mathcal{H}(\delta;1-z)}{\mathcal{H}(\delta;1)}\ .$$

To conclude the proof, note that:

$$\lim_{\delta\to 0}\frac{\mathcal{H}(\delta;1-z)}{\mathcal{H}(\delta;1)}=\lim_{\delta\to 0}\frac{(\mathcal{H}(\delta;1-z)-\mathcal{H}(0;1-z))/\delta}{(\mathcal{H}(\delta;1)-\mathcal{H}(0;1))/\delta}=\frac{\frac{\partial\mathcal{H}}{\partial t}(0;1-z)}{\frac{\partial\mathcal{H}}{\partial t}(0;1)}\ .\quad\blacksquare$$

Equations (53) and (54): Heavy-Tailed Rates

Proof. We shall make use of Karamata's Tauberian theorem (see, for example, corollary 8.1.7 in ref. 33) which asserts that $\Lambda(t)$ is heavy-tailed of order α (recall Eq. (51)) if and only if:

$$1-\mathbf{E}[\exp\{-\omega\Lambda(t)\}]\underset{\omega\to 0}{\sim}\omega^{\alpha}l(1/\omega)\ . \tag{A7}$$

Time-to-Pulsation

Using Eqs. (40) (the distribution of the time-to-pulsation), (36) (the definition of the function $\mathcal{H}(t;\theta)$), (52) (the regularity assumption), and (A7), we obtain that the short-time behavior of the time-to-pulsation is given by:

$$\begin{aligned}\mathbf{P}(\tau\leqslant t) &= 1-\exp\{-\mathcal{H}(t;1)\}\\ &= 1-\mathbf{E}\left[\exp\left\{-\int_0^t \Lambda(u)du\right\}\right]\\ &\underset{t\to 0}{\sim} 1-\mathbf{E}[\exp\{-\Lambda(0)t\}] \underset{t\to 0}{\sim} t^\alpha l(1/t)\ .\end{aligned}$$

Pulsation Multiplicity

From the proof of Eq. (42) (regarding the pulsation multiplicity) we have:

$$\mathbf{E}[z^M]=1-\lim_{\delta\to 0}\frac{1-\exp\{-\mathcal{H}(\delta;1-z)\}}{1-\exp\{-\mathcal{H}(\delta;1)\}}\ .$$

Now, using Eqs. (36) (the definition of the function $\mathcal{H}(t;\theta)$), (52) (the regularity assumption), and (A7), we obtain that:

$$\begin{aligned}\frac{1-\exp\{-\mathcal{H}(\delta;1-z)\}}{1-\exp\{-\mathcal{H}(\delta;1)\}} &= \frac{1-\mathbf{E}\left[\exp\left\{-(1-z)\int_0^\delta \Lambda(u)du\right\}\right]}{1-\mathbf{E}\left[\exp\left\{-\int_0^\delta \Lambda(u)du\right\}\right]}\\ &\underset{\delta\to 0}{\sim} \frac{1-\mathbf{E}[\exp\{-(1-z)\Lambda(0)\delta\}]}{1-\mathbf{E}[\exp\{-\Lambda(0)\delta\}]} \underset{\delta\to 0}{\sim} \frac{(1-z)^\alpha\delta^\alpha l\left(\frac{1}{(1-z)\delta}\right)}{\delta^\alpha l\left(\frac{1}{\delta}\right)}\\ &= (1-z)^\alpha\frac{l\left(\frac{1}{(1-z)\delta}\right)}{l\left(\frac{1}{\delta}\right)}\ .\end{aligned}$$

Finally, since the function $l(\cdot)$ is slowly varying at infinity we have

$$\lim_{\delta\to 0}\frac{l\left(\frac{1}{(1-z)\delta}\right)}{l\left(\frac{1}{\delta}\right)}=1\ ,$$

which, in turn, implies that

$$\mathbf{E}[z^M]=1-(1-z)^\alpha\ . \quad \blacksquare$$

A.4. Proofs: Section 5

Equation (58): Ornstein–Uhlenbeck Rates

Proof. First, using the explicit solution of the Ornstein–Uhlenbeck dynamics (Eq. (56)), we have:

$$\begin{aligned}\int_0^t \Lambda(u)du &= \int_0^t \left(\exp\{-\kappa u\}\Lambda(0) + \int_0^u \exp\{-\kappa(u-s)\}X(ds)\right)du \\ &= \left(\int_0^t \exp\{-\kappa u\}du\right)\Lambda(0) + \int_0^t \left(\int_s^t \exp\{-\kappa(u-s)\}du\right)X(ds) \\ &= K(t)\Lambda(0) + \int_0^t K(t-s)X(ds)\ , \end{aligned} \tag{A8}$$

where $K(t)=\kappa^{-1}(1-\exp\{-\kappa t\})$, $t\geqslant 0$.

Hence, using equation (A8), the independence of the initial value $\Lambda(0)$ and the Lévy driver X, and the Laplace transform of the equilibrium distribution of the Ornstein–Uhlenbeck rate process (Eq. (57)), we obtain that:

$$\begin{aligned}&\mathbf{E}\left[\exp\left\{-\theta\int_0^t \Lambda(u)du\right\}\right] \\ &=\mathbf{E}\left[\exp\left\{-\theta K(t)\Lambda(0)\right\}\right]\mathbf{E}\left[\exp\left\{-\int_0^t \theta K(t-s)X(ds)\right\}\right] \\ &=\exp\left\{-\tfrac{1}{\kappa}\int_0^{\theta K(t)}\tfrac{\phi(u)}{u}du\right\}\exp\left\{-\int_0^t \phi\left(\theta K(t-s)\right)ds\right\} \\ &=\exp\left\{-\tfrac{1}{\kappa}\int_0^{\theta K(t)}\tfrac{\phi(u)}{u}du\right\}\exp\left\{-\int_0^t \phi\left(\theta K(u)\right)du\right\} \\ &=\exp\left\{-\int_0^t\left(\tfrac{\phi(\theta K(u))}{\theta K(u)}\tfrac{K'(u)}{\kappa}+\phi\left(\theta K(u)\right)\right)du\right\}\ .\end{aligned}$$

To conclude the proof, note that:

$$\frac{\phi(\theta K(u))}{\theta K(u)}\frac{K'(u)}{\kappa}+\phi\left(\theta K(u)\right)=\frac{\phi\left(\frac{\theta}{\kappa}(1-\exp\{-\kappa u\})\right)}{1-\exp\{-\kappa u\}}\ . \qquad \blacksquare$$

A.5. Proofs: Section 6

Equation (68): Probability Tails of the Cumulative Pulsation

Proof. First, note that for all $t > 0$ we have

$$\mathcal{H}(t;\theta) \underset{\theta\to 0}{\sim} \mathcal{F}_\alpha(t)\cdot\phi(\theta) \ , \tag{A9}$$

where $\mathcal{F}_\alpha(t)$ is defined in equation (67).

Now, using the PGF of the cumulative pulsation (Eq. (37)), the asymptotic behavior of the Lévy characteristic $\phi(\cdot)$ at the origin (Eq. (66)), and Eq. (A9), we have:

$$\begin{aligned}1-\mathbf{E}[\exp\{-\omega Y(t)\}] &= 1-\exp\{-\mathcal{H}(t;1-\exp\{-\omega\})\}\\ &\underset{\omega\to 0}{\sim} \mathcal{H}(t;1-\exp\{-\omega\}) \underset{\omega\to 0}{\sim} \mathcal{F}_\alpha(t)\cdot\phi(1-\exp\{-\omega\})\\ &\underset{\omega\to 0}{\sim} \omega^\alpha\cdot\left(\mathcal{F}_\alpha(t)\cdot l\left(\tfrac{1}{1-\exp\{-\omega\}}\right)\right) \ .\end{aligned}$$

This, in turn (using Karamata's Tauberian theorem), implies that:

$$\mathbf{P}(Y(t)>y) \underset{y\to\infty}{\sim} \frac{\mathcal{F}_\alpha(t)}{\Gamma(1-\alpha)}\frac{l\left(\tfrac{1}{1-\exp\{-1/y\}}\right)}{y^\alpha} \ .$$

To conclude the proof, note that since $l(\cdot)$ is a slowly varying function at infinity we have

$$l\left(\tfrac{1}{1-\exp\{-1/y\}}\right) \underset{y\to\infty}{\sim} l(1/y) \ . \quad \blacksquare$$

Equation (69): Auto-Correlation

Proof. For all $b > a > 0$ we have (σ^2 denotes the variance of the Lévy driver X):

$$\begin{aligned}&\mathbf{Cov}(X_F(b), X_F(a))\\ &= \mathbf{Cov}\left(\int_{-\infty}^{\infty}(F(b-u)-F(-u))\,X(du), \int_{-\infty}^{\infty}(F(a-v)-F(-v))\,X(dv)\right)\\ &= \sigma^2\int_{-\infty}^{\infty}[F(b-u)-F(-u)][F(a-u)-F(-u)]\,du\\ &= \sigma^2\int_{-\infty}^{\infty}[F(u+b)-F(u)][F(u+a)-F(u)]\,du.\end{aligned} \tag{A10}$$

On the other hand, using conditioning, we obtain that:

$$\begin{aligned}
&\mathbf{Cov}\,(Y(T+\Delta)-Y(T),Y(\Delta))\\
&\quad=\mathbf{Cov}\,(N(X_F(T+\Delta))-N(X_F(T)),N(X_F(\Delta)))\\
&\quad=\mathbf{E}\,[\mathbf{Cov}\,(N(X_F(T+\Delta))-N(X_F(T)),N(X_F(\Delta))|X_F)]\\
&\qquad+\mathbf{Cov}\,(\mathbf{E}\,[N(X_F(T+\Delta))-N(X_F(T))|X_F],\mathbf{E}\,[N(X_F(\Delta))|X_F])\\
&\quad=0+\mathbf{Cov}\,(X_F(T+\Delta)-X_F(T),X_F(\Delta))\\
&\quad=\mathbf{Cov}\,(X_F(T+\Delta),X_F(\Delta))-\mathbf{Cov}\,(X_F(T),X_F(\Delta))\,.
\end{aligned}\tag{A11}$$

Hence, combining Eqs. (A10) and (A11) together, yields:

$$\begin{aligned}
&\rho_\Delta(T):=\mathbf{Cov}\,(Y(T+\Delta)-Y(T),Y(\Delta))\\
&\quad=\sigma^2\int_{-\infty}^{\infty}[F(u+T+\Delta)-F(u+T)][F(u+\Delta)-F(u)]\,du\\
&\quad=\sigma^2\int_0^{\infty}[F(u+T)-F(u+T-\Delta)][F(u)-F(u-\Delta)]\,du. \quad \blacksquare
\end{aligned}$$

ACKNOWLEDGMENTS

The authors wish to: (i) acknowledge *Ariel Lubelski* for creating the simulations appearing in the Figures; and, (ii) thank an anonymous referee for a meticulous reading of the manuscript – which resulted in a long list of most constructive comments and suggestions.

REFERENCES

1. Y. J. Jung, E. Barkai, and R. J. Silbey, *Chem. Phys.* **284**:181 (2002).
2. H. Yang et. al., *Science* **302**:262 (2003).
3. M. Orrit, *Science* **302**:239 (2003).
4. A. Meller, *J. Phys. Cond. Matt.* **15**: 581 (2003).
5. R. D. Reiss, *A Course on Point Processes* (Springer-Verlag, 1993).
6. P. Lévy, *Calcul des Probabilités* (Gauthier-Villars, 1925).
7. P. Lévy, *Théorie de L'Addition des Variables Aléatoires* (Gauthier-Villars, 1954).
8. P. Lévy, *Processus Stochastiques et Mouvement Brownien* (Gauthier-Villars, 1965).
9. V.M. Zolotarev, *One-Dimensional Stable Distributions* (AMS, 1986).
10. G. Samrodintsky and M. S. Taqqu, *Stable non-Gaussian Random Processes* (CRC Press, 1994).
11. J. Bertoin, *Lévy Processes* (Cambridge University Press, 1996); J. Bertoin, *Subordinators: examples and applications, Lecture Notes in Mathematics* 1717 (Springer, 1999).
12. K. Sato, *Lévy Processes and Infinitely Divisible Distributions* (Cambridge University Press, 1999).
13. V. V. Uchaikin and V. M. Zolotarev, *Chance and Stability, Stable Distributions and Their Applications* (V.S.P. Intl. Science, 1999).

14. O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnic eds., *Lévy Processes* (Birkhauser, 2001).
15. J. P. Bouchaud and A. Georges, *Phys. Rep.* **195**: 12 (1990).
16. M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, *Nature* **363**:31 (1993).
17. M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch eds., *Lévy Flights and Related Topics* (Springer, 1995).
18. J. Klafter, M. F. Shlesinger, and G. Zumofen, *Phys. Today* **49**:(2) 33 (1996).
19. G. M. Viswanathan, *et al.*, *Nature* **401**:911 (1999); G. M. Viswanathan, *et al.*, *Physica A* **282**:1 (2000); G. M. Viswanathan, *et al.*, *Physica A* **295**:85 (2001).
20. G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, 1994).
21. B. D. Hughes, *Random Walks and Random Environments* (Oxford University Press, 1995).
22. R. Metzler, J. Klafter, *Phys. Rep.* **339**:1 (2000).
23. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed (Wiley, 1971).
24. I. M. Sokolov, *Phys. Rev. E* **63** 011104 (2000); I. M. Sokolov, *Phys. Rev. E* **66**:041101 (2002).
25. A. A. Stanislavsky, *Theor. Appl. Math. Phys.* **138**(3):481 (2004).
26. I. Eliazar, J. Klafter, *Physica D* **187**:30 (2004).
27. I. Eliazar and J. Klafter, to appear in *J. Stat. Phys.* (titled: "Lévy, Ornstein-Uhlenbeck, and Subordination: Spectral vs Jump description").
28. D. R. Cox, *J. R. Stat. Soci. B* **17**:129 (1955).
29. J. F. C. Kingman, *Poisson Processes* (Oxford University Press, 1993).
30. J. Grandell, *Doubly Stochastic Poisson Processes* (Springer, 1976).
31. D. R. Cox, *Renewal Theory* (Methuen (London), 1962).
32. P. Embrechts and M. Maejima, *Selfsimilar Processes* (Princeton University Press, 2002).
33. N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation* (Cambridge University Press, 1987).
34. B. B. Mandelbrot, *Multifractals and* $1/f$ *Noise* (Springer, 1999).
35. H. C. Tijms, *Stochastic Models: An Algorithmic Approach* (Wiley, 1995).
36. E. Barlow and F. Proschan, *Mathematical Theory of Reliability (Classics in Appplied Mathematics 17, reprint edition)* (Society for Industrial & Appplied Mathematics, 1996).
37. T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley-Interscience, 1991).
38. I. Eliazar and J. Klafter, *J. Stat. Phys.* **111**(314):739 (2003).
39. J. M. Chambers, C. L. Mallows, and B. Stuck, *J. Amer. Stat. Assoc.* **71**:340 (1976).